PERGAMON

International Journal of Solids and Structures 40 (2003) 5423–5439

INTERNATIONAL JOURNAL OF SOLIDS and STRUCTURES

www.elsevier.com/locate/ijsolstr

Comparison of dynamic response of structures with uncertain-but-bounded parameters using non-probabilistic interval analysis method and probabilistic approach

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Received 17 July 2002; received in revised form 9 April 2003

Abstract

Now by combining the finite element analysis and interval mathematics, a new, non-probabilistic, set-theoretical models, that is interval analysis method is being developed in scientific and engineering communities to predict the variability or uncertainty resulting from the unavoidable scatter in structural parameters and the external excitations as an alternative to the classical probabilistic approaches. Interval analysis methods of uncertainty were developed for modeling uncertain parameters of structures, in which bounds on the magnitude of uncertain parameters are only required, not necessarily knowing the probabilistic distribution densities. Instead of conventional optimization studies, where the minimum possible response is sought, here an uncertainty modeling is developed as an anti-optimization problem of finding the least favorable response and the most favorable response under the constraints within the set-theoretical description. In this study, we presented the non-probabilistic interval analysis method for the dynamical response of structures with uncertain-but-bounded parameters. Under the condition of the interval vector, which contains the uncertain-but-bounded parameters, determined from probabilistic statistical information or stochastic sample test, through comparing between the non-probabilistic interval analysis method and the probabilistic approach in the mathematical proof and the numerical examples, we can see that the region of the dynamical response of structures with uncertain-but-bounded parameters obtained by the interval analysis method contains that produced by the probabilistic approach. In other words, the width of the maximum or upper and minimum or lower bounds on the dynamical responses yielded by the probabilistic approach is tighter than those produced by the interval analysis method. This kind of results is coincident with the meaning of the probabilistic theory and interval mathematics.

Keywords: Dynamical response; Finite element analysis; Non-probabilistic interval analysis method; Uncertain-but-bounded parameters; Probabilistic approach

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doi:10.1016/S0020-7683(03)00282-8

1. Introduction

The treatment of external excitations and structural parameters as uncertain variables has been the subject of some scientists and engineers for many years (Ibrahim, 1987). The concept of uncertainty plays an important role in the investigation of various science and engineering problems. In structural static and dynamical analysis, the structural external excitations and parameters are subject to variation under the influence of many factors such as fluctuations in the stiffness, damping and mass matrices caused by uncertain variations in material properties, uncertainty in boundary conditions, and variations caused by manufacturing and assembly techniques. The mechanical properties of structural systems are subject to a certain degree of uncertainty because the structural properties of their finite elements are not measured or evaluated exactly. The uncertainties in structural systems affect to a large extent their design and operating performance. According to conventional structural analysis procedures, these external excitations and structural parameters should be modeled as random variables or processed with a probability distribution representing the distribution of the measured values. This modeling results in random response of structural systems in question. In current literatures of structural response problems with random uncertainties, there are three main ways (Li and Liao, 2001), they are: the stochastic finite element method, the Monte Carlo simulation method and the orthogonal series expansion method. The Monte Carlo simulation method (Astill et al., 1972; Wall and Bucher, 1987) is very efficient in this aspect of structural random analysis, but it is quite time-consuming. The stochastic finite element method (Collins and Thompson, 1969; Zhu and Wu, 1991; Kleiber and Hien, 1992; Liu et al., 1985) is very powerful in solving the random eigenvalue problem, static analysis problem and structural stability problem, but the method is haunted by the notorious secular term in structural random dynamical response analysis. In the orthogonal series expansion method (Sun, 1979; Ghanem and Spanos, 1990), the structural response may be expanded an orthogonal series and the corresponding numerical characteristics are given as analytical solution form. Despite the success of the above probabilistic analysis approaches, one may recognize that uncertainties in structures can be modeled on the basis of alternative, non-probabilistic conceptual frameworks. In the frequently encountered case where the sufficient knowledge about the external excitations and structural parameters are absent for substantiation of the stochastic analysis, based on convex analysis and interval mathematics, in recent studies by Ben-Haim and Elishakoff (1990), Elishakoff et al. (1994a,b), Qiu and Elishakoff (1998), Qiu et al. (1995, 1996, 2001a,b), Chen and Yang (2000), Mullen and Muhanna (1999), and Pantelides and Ganerli (2001), Ganzerli and Pantelides (2000), convex models and interval analysis methods of uncertainty were developed for modeling the structural uncertain external excitations and parameters, in which bounds on the magnitude of uncertain external excitations and parameters are only required, not necessarily knowing the probabilistic distribution densities, following the general methodologies developed in the monographs. It was assumed that the structural characteristics fall into the multidimensional ellipsoid or solid ball, instead of conventional optimization studies, where the minimum possible response is sought, here an uncertainty modeling is developed as an anti-optimization problem of finding the least favorable response and the most favorable response under the constrains within the set-theoretical description. Convex (ellipsoidal or interval) sets have been used for modeling uncertain phenomena in a wide range of engineering applications. However, most formulations of convex models and interval analysis methods were given in terms of the analytic form (Ben-Haim and Elishakoff, 1990); the analytic approaches are not convenient for dealing with the uncertain problem in practical engineering. In the study, in virtue of finite element analysis, the anti-optimum numerical methods—the numerical interval analysis method for the dynamical response of structures with uncertain-but-bounded external excitations and parameters is presented. By means of the mathematical proof and numerical examples, the numerical interval analysis method and the probabilistic models of the structural dynamical response are critically contrasted. In these comparisons, we can see that the region of the dynamical response of structures with uncertain-but-bounded parameters calculated by the interval analysis method includes that obtained by the probabilistic approach. That is to say, the width of the maximum or upper and minimum or lower bounds on the dynamical response produced by the interval analysis method is larger than those yielded by the probabilistic approach.

2. Problem statement

Consider the equation of motion (Meirovotch, 1980; Weaver and Johnston, 1987) of a general dynamical system with n degree of freedom in the following form:

$$M\ddot{x}(t) + C\dot{x}(t) + Kx(t) = F(t) \tag{1}$$

where $M = (m_{ij})$, $C = (c_{ij})$ and $K = (k_{ij})$ are the mass, damping, and stiffness matrices; $F(t) = (f_i(t))$ is the external load vector. $x(t) = (x_i(t))$, $\dot{x}(t) = (\dot{x}_i(t))$ and $\ddot{x}(t) = (\ddot{x}_i(t))$ are the displacement, velocity, and acceleration vectors of the finite element assemblage. The matrix $M = (m_{ij})$ is the positive definite, $C = (c_{ij})$ and $K = (k_{ij})$ are the positive semi-definite matrices.

By finite element analysis, we know that the mass matrix $M = (m_{ij})$, the damping matrix $C = (c_{ij})$, the stiffness matrix $K = (k_{ij})$ and the external load vector $F(t) = (f_i(t))$ depend on the structural parameter vector $a = (a_i)$ and may be expressed as functions of the structural parameter vector $a = (a_i)$, i.e.

$$\begin{aligned} &M = M(a) = (m_{ij}(a)), \qquad C = C(a) = (c_{ij}(a)) \\ &K = K(a) = (k_{ij}(a)), \qquad F(t) = F(a,t) = (f_i(a,t)) \end{aligned} \tag{2}$$

in which $a = (a_i)$ is m-dimensional vector. Thus, Eq. (1) can be rewritten as

$$M(a)\ddot{x}(a,t) + C(a)\dot{x}(a,t) + K(a)x(a,t) = F(a,t) \tag{3}$$

Consider a realistic situation in which available information on the structural parameter vector $a = (a_i)$ is not enough to justify an assumption on its probabilistic characteristics, we follow the thought of interval mathematics or interval analysis (Moore, 1979; Alefeld and Herzberger, 1983) and assume that the structural parameter vector $a = (a_i)$ belong to a bounded convex set—interval vector

$$a \in a^I = [\underline{a}, \overline{a}] = (a_i^I), \quad a_i \in a_i^I = [\underline{a}_i, \overline{a}_i], \qquad i = 1, 2, \ldots, m \tag{4}$$

where $\overline{a} = (\overline{a}_i)$ and $\underline{a} = (\underline{a}_i)$ are the upper and lower bounds of structural parameters $a = (a_i)$, respectively. From interval mathematics, we know that Eq. (3) describes a "box" with m order of dimension.

Suppose that the upper bound vector $\overline{a} = (\overline{a}_i)$ and the lower bound vector $\underline{a} = (\underline{a}_i)$ of the structural parameter vector $a = (a_i)$ are given, the object is to find all the possible dynamical responses $x(t)$ satisfying the dynamical equation (3), where a is assumed all possible values inside the interval parameter vector a^I. This infinite number of dynamical responses constitutes a bounded response set

$$\Gamma = \left\{ x(a,t) : M(a)\ddot{x}(a,t) + C(a)\dot{x}(a,t) + K(a)x(a,t) = F(a,t), a \in a^I \right\} \tag{5}$$

In general, the set Γ has a very complicated region.

In interval mathematics (Moore, 1979; Alefeld and Herzberger, 1983), solving the dynamical response problem (3) subject to (4) is synonymous to finding a multi-dimensional rectangle or interval vector containing dynamical response set (5) for the interval structural parameter vector. In other words, we seek the upper and lower bounds (or interval dynamical response vector) on the dynamical response set (5), i.e.

$$x^I(a,t) = [\underline{x}(a,t), \overline{x}(a,t)] = (x_i^I(a,t)) \tag{6a}$$

or component form

$$x_i^I(a,t) = [\underline{x}_i(a,t), \overline{x}_i(a,t)], \quad i = 1, 2, \ldots, n \tag{6b}$$

where $\overline{x}(a,t) = (\overline{x}_i(a,t))$ and $\underline{x}(a,t) = (\underline{x}_i(a,t))$, and

$$\overline{x}(a,t) = \max\left\{x(a,t) : x(a,t) \in R^n, M(a)\ddot{x}(a,t) + C(a)\dot{x}(a,t) + K(a)x(a,t) = F(a,t), a \in a^I\right\} \tag{7}$$

and

$$\overline{x}(a,t) = \min\left\{x(a,t) : x(a,t) \in R^n, M(a)\ddot{x}(a,t) + C(a)\dot{x}(a,t) + K(a)x(a,t) = F(a,t), a \in a^I\right\} \tag{8}$$

In the sequel, our aim is to determine the upper and lower bounds of the interval dynamical response.

3. Interval analysis method

In this section, we will calculate the interval dynamical response vector of structures with uncertain-but-bounded parameters making use of interval mathematics.

By means of Eq. (4), we may define the nominal value vector or midpoint vector (Moore, 1979; Alefeld and Herzberger, 1983) of the interval structural parameter vector as

$$a^c = (a_i^c) = m(a^I) = \frac{(\overline{a} + \underline{a})}{2}, \quad a_i^c = m(a_i^I) = \frac{(\overline{a}_i + \underline{a}_i)}{2}, \qquad i = 1, 2, \ldots, m \tag{9}$$

and the deviation amplitude vector or the uncertain radius vector of the interval structural parameter vector as

$$\Delta a = (\Delta a_i) = \mathrm{rad}(a^I) = \frac{(\overline{a} - \underline{a})}{2}, \quad \Delta a_i = \mathrm{rad}(a_i^I) = \frac{(\overline{a}_i - \underline{a}_i)}{2}, \qquad i = 1, 2, \ldots, m \tag{10}$$

Thus, based on interval mathematics, the interval structural parameter vector is decomposed into the sum of the nominal value vector and the deviation vector, i.e.

$$a^I = [\underline{a}, \overline{a}] = [a^c - \Delta a, a^c + \Delta a] = [a^c, a^c] + [-\Delta a, \Delta a] = a^c + \Delta a^I = a^c + \Delta a[-1, 1] = a^c + \Delta a e_\Delta \tag{11}$$

where $\overline{a} = a^c + \Delta a$, $\underline{a} = a^c - \Delta a$, $\Delta a^I = [-\Delta a, \Delta a]$, $e_\Delta = [-1, 1]$.

In terms of the expression (11), the interval structural parameter vector may be written in the following form:

$$a = a^c + \delta a, \quad |\delta a| \leqslant \Delta a \tag{12a}$$

or component form

$$a_i = a_i^c + \delta a_i, \quad |\delta a_i| \leqslant \Delta a_i, \;\; i = 1, 2, \ldots, m \tag{12b}$$

Using Taylor series the dynamical response $x_i(a,t)$, $i = 1, 2, \ldots, n$ about a^c is developed as

$$x_i(a,t) = x_i(a^c + \delta, t) = x_i(a^c, t) + \sum_{j=1}^{m} \frac{\partial x_i(a^c, t)}{\partial a_j} \delta a_j \tag{13}$$

in which

$$\delta a_j \in \Delta a_j^I = [-\Delta a_j, \Delta a_j], \quad j = 1, 2, \ldots, m \tag{14}$$

By making use of the interval extension in interval mathematics, from the expression (13), we can obtain the interval extension of the dynamical response of structures

$$x_i^I(a,t) = x_i(a^c, t) + \sum_{j=1}^{m} \left| \frac{\partial x_i(a^c, t)}{\partial a_j} \right| \Delta a_j^I \tag{15}$$

After the interval operations, from the above equation, we have

$$\overline{x}_i(a,t) = x_i(a^c,t) + \sum_{j=1}^{m} \left| \frac{\partial x_i(a^c,t)}{\partial a_j} \right| \Delta a_j, \quad i = 1,2,\ldots,n \tag{16}$$

and

$$\underline{x}_i(a,t) = x_i(a^c,t) - \sum_{j=1}^{m} \left| \frac{\partial x_i(a^c,t)}{\partial a_j} \right| \Delta a_j, \quad i = 1,2,\ldots,n \tag{17}$$

By Eqs. (16) and (17) we can determine the interval region of the dynamical response of structures with uncertain-but-bounded parameters using the interval analysis method. Obviously, if the approximation of Taylor series extension is omitted, according to the meaning of interval mathematics, the probability of the dynamical response of structures with uncertain-but-bounded parameters belonging to the interval region of the dynamical response is equal to unity.

4. Determining the first derivative of the dynamical response

Recall that the mass matrix $M(a)$, the damping matrix $C(a)$ and the stiffness matrix $K(a)$ are symmetric. Assume that all elements in $M(a)$, $C(a)$, $K(a)$ and all components in $F(a,t)$ are continuously differentiable with respect to the structural parameter a. The implicit function theorem thus guarantees that the dynamical response or solution $x(t) = x(a,t)$ of Eq. (3) is also continuously differentiable.

Differentiating both sides of Eq. (3) with respect to a yields

$$M(a)\frac{\partial \ddot{x}(a,t)}{\partial a} + C(a)\frac{\partial \dot{x}(a,t)}{\partial a} + K(a)\frac{\partial x(a,t)}{\partial a} = R(a,t) \tag{18}$$

where

$$R(a,t) = \frac{\partial F(a,t)}{\partial a} - \frac{\partial M(a)}{\partial a}\ddot{x}(a,t) - \frac{\partial C(a)}{\partial a}\dot{x}(a,t) - \frac{\partial K(a)}{\partial a}x(a,t) \tag{19}$$

Substitution of $a = a^c$ into Eq. (18) leads to the following equation:

$$M(a^c)\frac{\partial \ddot{x}(a^c,t)}{\partial a} + C(a^c)\frac{\partial \dot{x}(a^c,t)}{\partial a} + K(a^c)\frac{\partial x(a^c,t)}{\partial a} = R(a^c,t) \tag{20}$$

in which

$$R(a^c,t) = \frac{\partial F(a^c,t)}{\partial a} - \frac{\partial M(a^c)}{\partial a}\ddot{x}(a^c,t) - \frac{\partial C(a^c)}{\partial a}\dot{x}(a^c,t) - \frac{\partial K(a^c)}{\partial a}x(a^c,t) \tag{21}$$

In finite element analysis, the ith element mass matrix is denoted as $M_i(a)$, the ith element damping matrix as $C_i(a)$, the ith element stiffness matrix as $K_i(a)$, and the ith element nodal loads as $f_i(a,t)$. After mass, damping, stiffness and nodal loads for all the nodes of the element have been transformed to global directions, with the direct stiffness method, the global mass matrix $M(a)$, the global damping matrix $C(a)$, the global stiffness matrix $K(a)$ and nodal load vectors $F(a,t)$ are obtained by summing the element mass matrices, the element damping matrices, the element stiffness matrices and the nodal loads over all *NE* elements in the structure, to obtain

$$M(a) = \sum_{i=1}^{NE} M_i(a), \qquad C(a) = \sum_{i=1}^{NE} C_i(a), \qquad K(a) = \sum_{i=1}^{NE} K_i(a), \qquad F(a,t) = \sum_{j=1}^{NE} f_j(a,t) \tag{22}$$

Using Eq. (22) the first derivative required in Eq. (20) may be written as the sum of derivatives of element matrices as

$$\frac{\partial M(a)}{\partial a} = \frac{\partial}{\partial a}\left(\sum_{i=1}^{NE} M_i(a)\right) = \sum_{i=1}^{NE} \frac{\partial M_i(a)}{\partial a}, \qquad \frac{\partial C(a)}{\partial a} = \frac{\partial}{\partial a}\left(\sum_{i=1}^{NE} C_i(a)\right) = \sum_{i=1}^{NE} \frac{\partial C_i(a)}{\partial a} \tag{23a}$$

$$\frac{\partial K(a)}{\partial a} = \frac{\partial}{\partial a}\left(\sum_{i=1}^{NE} K_i(a)\right) = \sum_{i=1}^{NE} \frac{\partial K_i(a)}{\partial a}, \qquad \frac{\partial F(a,t)}{\partial a} = \sum_{j=1}^{NE} \frac{\partial f_j(a,t)}{\partial a} \tag{23b}$$

The practicality of this computation follows from at least in two cases: (1) for each element mass matrix, each element damping matrix, each element stiffness matrix and each nodal external load vector, the matrices $M_i(a)$, $C_i(a)$, $K_i(a)$ and the vector $f_i(a,t)$ will depend on only a small number of structural parameters that are associated with the given element and its nodes. Thus, only a few terms in the sum of Eqs. (23) will be different from zero. (2) Evaluation of derivatives of the element bilinear forms in Eqs. (23) requires calculation of only a moderate number of terms. There are important practical considerations in adapting programmes for computation of derivatives that are required in structural sensitivity analysis.

Another practical consideration that should not be overlooked involves calculating derivatives of the element mass matrices, the element mass matrices, the element damping matrices, the element stiffness matrices and the element nodal load vectors that are implicitly generated. Many modern finite element formulations carry out numerical integration to evaluate the element mass matrices, the element damping matrices, the element stiffness matrices and the element nodal load vectors, rather than using closed from expressions in terms of structural parameters. For implicitly generated the element mass matrices, the element damping matrices, the element stiffness matrices and the element nodal load vectors, the differentiation can be carried through the sequence of calculations used to generate the element mass matrices, the element damping matrices, the element stiffness matrices and the element nodal load vectors, thus leading to implicit first derivative routines. An alternative approach is simply to perturb one structural parameter at a time and use finite difference to approximate element matrix and component vector derivative. For example

$$\frac{\partial M_i(a)}{\partial a} \approx \frac{M_i(a+\delta a) - M_i(a)}{\delta a}, \qquad \frac{\partial C_i(a)}{\partial a} \approx \frac{C_i(a+\delta a) - C_i(a)}{\delta a} \tag{24a}$$

$$\frac{\partial K_i(a)}{\partial a} \approx \frac{K_i(a+\delta a) - K_i(a)}{\delta a}, \qquad \frac{\partial f_i(a,t)}{\partial a} \approx \frac{f_i(a+\delta a,t) - f_i(a,t)}{\delta a} \tag{24b}$$

where δa is a small perturbation in the structural parameter a.

5. Probabilistic approach

In this section, we will determine the interval dynamical response of structures with uncertain-but-bounded parameters by probabilistic approach.

Assume that the m-dimensional uncertain structural parameter vector $a = (a_i)$ is random variable (Elishakoff, 1983). Thus, the dynamical response $x(a,t)$ is also random. If we denote the random structural parameter vector's expected value, or the mean value (MV), as

$$E(a) = (E(a_i)) = a^E = (a_i^E) \tag{25}$$

then Eq. (13) can be interpreted as a Taylor's series expansion of the random dynamical response about the mean value $x_i(a^E,t)$, $i = 1,2,\ldots,n$ of the random structural parameter vector $a = (a_i)$.

For the random structural parameter vector $a = (a_i)$, the variance is defined by

$$\mathrm{Var}(a) = (\mathrm{Var}(a_i)) = D(a) = (D(a_i)) \tag{26}$$

Then the standard deviation of the random structural parameter vector $a = (a_i)$ is defined as

$$\sigma(a) = (\sigma(a_i)) = \sqrt{\mathrm{Var}(a)} = \sqrt{D(a)} = (\mathrm{Var}(a_i)) = (\sqrt{D(a_i)}) \tag{27}$$

The mean value or expected value of the dynamical response is obtained by taking the expected value of both side of Eq. (13). In so doing, it follows that

$$E\{x_i(a,t)\} = E\{x_i(a^E,t)\} + E\left(\sum_{j=1}^{m} \frac{\partial x_i(a^E,t)}{\partial a_j}\delta a_j\right) = x_i(a^E,t) + \sum_{j=1}^{m} \frac{\partial x_i(a^E,t)}{\partial a_j} E(a_j - a_j^E), \quad i = 1,2,\ldots,n \tag{28}$$

and nothing that the term $E(\delta a_j) = E(a_j - a_j^E)$ is zero, we obtain

$$E\{x_i(a,t)\} = x_i(a^E,t), \quad i = 1,2,\ldots,n \tag{29}$$

For the variance of the dynamical response $x_i(a,t)$ we obtain in a similar way as follows:

$$\mathrm{Var}(x_i(a,t)) = D(x_i(a,t)) = \sum_{j=1}^{m} \left(\frac{\partial x_i(a^E,t)}{\partial a_j}\right)^2 D(a_j) + \sum_{k=1}^{m}\sum_{l=1}^{m} \frac{\partial x_i(a^E,t)}{\partial a_k}\frac{\partial x_i(a^E,t)}{\partial a_l}\mathrm{Cov}(a_k,a_l) \tag{30}$$

where $\mathrm{Cov}(a_k,a_l)$ is the covariance of the random structural parameter variables and is defined as

$$\mathrm{Cov}(x_k,x_l) = E[(x_k(a,t) - E[x_k(a,t)])(x_l(a,t) - E[x_l(a,t)])] \tag{31}$$

When the random structural parameter variables are independent, the variance of the dynamical response can be reduced as

$$\mathrm{Var}(x_i(a,t)) = D(x_i(a,t)) = \sum_{j=1}^{m} \left(\frac{\partial x_i(a^E,t)}{\partial a_j}\right)^2 D(a_j) = \sum_{j=1}^{m} \left(\frac{\partial x_i(a^E,t)}{\partial a_j}\sigma(a_j)\right)^2 = \sum_{j=1}^{m} \left(\frac{\partial x_i(a^E,t)}{\partial a_j}\sigma_j\right)^2 \tag{32}$$

Obviously, the standard deviation of the dynamical response $x_i(a,t)$ is

$$\sigma(x_i(a,t)) = \sqrt{D(x_i(a,t))} = \sqrt{\sum_{j=1}^{m} \left(\frac{\partial x_i(a^E,t)}{\partial a_j}\sigma_j\right)^2} \tag{33}$$

Thus, let k be a positive integer, the probabilistic region of k times standard deviations of its mean value of the random dynamical response is

$$y_i^I = [\underline{y}_i(a,t), \overline{y}_i(a,t)] = [x_i(a^E,t) - k\sigma(x_i(a,t), x_i(a^E,t) + k\sigma(x_i(a,t)], \quad i = 1,2,\ldots,n \tag{34}$$

where the probabilistic upper bound is

$$\overline{y}_i(a,t) = x_i(a^E,t) + k\sigma(x_i(a,t)) = x_i(a^E,t) + k\sqrt{\sum_{j=1}^{m} \left(\frac{\partial x_i(a^E,t)}{\partial a_j}\sigma_j\right)^2}, \quad i = 1,2,\ldots,n \tag{35}$$

and the probabilistic lower bound is

$$\underline{y}_i(a,t) = x_i(a^E,t) - k\sigma(x_i(a,t)) = x_i(a^E,t) - k\sqrt{\sum_{j=1}^{m}\left(\frac{\partial x_i(a^E,t)}{\partial a_j}\sigma_j\right)^2}, \quad i = 1,2,\ldots,n \tag{36}$$

By Eqs. (35) and (36) we can obtain the probabilistic region of the dynamical response of structures with uncertain-but-bounded parameters using the probabilistic approach. According to the Tchebycheff's inequality, we know that the probability of the random variable with finite variance falling within k standard deviations of its mean is at least $1 - 1/k^2$, and the bound is independent of the distribution of the random variable, provided it has a finite variance. For sufficient large k, in the numerical example, when using the probabilistic approach to estimate the upper and lower bound of structural response, the value of k times standard deviations in Eqs. (35) and (36) will result in almost a certain event.

6. Comparison of non-probabilistic interval analysis method and probabilistic approach

For convenience of comparison, we first prove the following inequality.

For any real m-tuples $a_i \geqslant 0, 1, 2, \ldots, m$, then the following inequality holds:

$$\sum_{i=1}^{m} a_i \geqslant \sqrt{\sum_{i=1}^{m} a_i^2} \tag{37}$$

Proof. Adding $\sum_{i=1}^{m} a_i^2$ to the both sides of the following inequality:

$$2\sum_{\substack{i,j=1\\ i\neq j}}^{m} a_i a_j \geqslant 0 \tag{38}$$

to arrive at

$$\sum_{i=1}^{m} a_i^2 + 2\sum_{\substack{i,j=1\\ i\neq j}}^{m} a_i a_j \geqslant \sum_{i=1}^{m} a_i^2 \tag{39}$$

arranging the above inequality to produce

$$\left(\sum_{i=1}^{m} a_i\right)^2 \geqslant \sum_{i=1}^{m} a_i^2 \tag{40}$$

taking the square of the inequality to yield

$$\sum_{i=1}^{m} a_i \geqslant \sqrt{\sum_{i=1}^{m} a_i^2} \qquad \square \tag{41}$$

Assume that we obtain the interval regions of the uncertain-but-bounded structural parameters based on the probabilistic statistical information or stochastic sample test and they can be expressed as the following interval vector form:

$$a^I = [\underline{a}, \overline{a}] = (a_i^I) = [a^E - k\sigma, a^E + k\sigma] \tag{42}$$

and its component form

$$a_i^I = [\underline{a}_i, \overline{a}_i] = \left[a_i^E - k\sigma_i, a_i^E + k\sigma_i\right], \quad i = 1,2,\ldots,m \tag{43}$$

where $\overline{a} = (\overline{a}_i)$, $\overline{a}_i = a_i^E - k\sigma_i$, $i = 1, 2, \ldots, m$, and $\underline{a} = (\underline{a}_i)$, $\underline{a}_i = a_i^E + k\sigma_i$, $i = 1, 2, \ldots, m$, are respectively the upper bound vector and the lower bound vector of the interval vector $a^I = [\underline{a}, \overline{a}] = (a_i^I)$, the vectors $a^E = (a_i^E)$ and $\sigma = \sigma(a) = (\sigma(a_i)) = (\sigma_i)$ are respectively the mean value and the standard deviation of the uncertain structural parameter vector $a = (a_i)$, and k is a positive integer. According to the Tchebycheff's inequality in probabilistic theory, we know that the probability of the uncertain structural parameter $a = (a_i)$ with finite variance $D = D(a) = (D_i) = (D(a_i))$ falling within k standard deviations $\sigma = \sqrt{D} = (\sigma_i) = (\sqrt{D_i})$ of its mathematical expectation is at least $1 - 1/k^2$, and the bound is independent of the distribution of the uncertain structural parameter, provided it has a finite variance. Obviously, from (42) and (43), the nominal value vector or midpoint vector of the uncertain structural parameter vector $a = (a_i)$ can be calculated as follows:

$$a^c = (a_i^c) = m(a^I) = a^E, \quad a_i^c = m(a_i^I) = a_i^E, \qquad i = 1, 2, \ldots, m \tag{44}$$

and the deviation amplitude vector or the uncertain radius vector of the uncertain structural parameter vector $a = (a_i)$ can be determined

$$\Delta a = (\Delta a_i) = \mathrm{rad}(a^I) = k\sigma, \quad \Delta a_i = \mathrm{rad}(a_i^I) = k\sigma_i, \qquad i = 1, 2, \ldots, m \tag{45}$$

Thus, in terms of the expressions (44) and (45), the interval region (16) and (17) of the structural dynamical response can be rewritten as

$$\overline{x}_i(a, t) = x_i(a^E, t) + \sum_{j=1}^{m} \left| \frac{\partial x_i(a^E, t)}{\partial a_j} \right| k\sigma_j, \quad i = 1, 2, \ldots, n \tag{46}$$

and

$$\underline{x}_i(a, t) = x_i(a^E, t) - \sum_{j=1}^{m} \left| \frac{\partial x_i(a^E, t)}{\partial a_j} \right| k\sigma_j, \quad i = 1, 2, \ldots, n \tag{47}$$

For the sum expression

$$\sum_{j=1}^{m} \left| \frac{\partial x_i(a^E, t)}{\partial a_j} \right| k\sigma_j,$$

by means of the inequality (41), we have that

$$\sum_{j=1}^{m} \left| \frac{\partial x_i(a^E, t)}{\partial a_j} k\sigma_j \right| \geqslant \sqrt{\sum_{j=1}^{m} \left| \left(\frac{\partial x_i(a^E, t)}{\partial a_j} k\sigma_j \right)^2 \right|} = k \sqrt{\sum_{j=1}^{m} \left(\frac{\partial x_i(a^E, t)}{\partial a_j} \sigma_j \right)^2} \tag{48}$$

Since the inequality (48), from Eqs. (35), (36), (46) and (47), we can deduce

$$\underline{x}_i(a, t) \leqslant \underline{y}_i(a, t) \leqslant \overline{y}_i(a, t) \leqslant \overline{x}_i(a, t) \tag{49}$$

The expression (49) means that under the condition of the interval vector of the uncertain parameters determined from the probabilistic information, the width of the dynamical response obtained by the interval analysis method is larger than that by the probabilistic approach for structures with uncertain-but-bounded structural parameters. Namely the lower bounds within interval analysis method are smaller than those predicted by the probabilistic approach, and the upper bounds furnished by the interval analysis method are larger than those yielded by the probabilistic approach. This is just the results which we hope, since according to the definition of probabilistic theory and interval mathematics, the region by determined by the interval analysis method should contain that predicted by the probabilistic approach.

7. Numerical examples

Example I. Fig. 1 shows a cantilever beam with 11 nodes, 10 elements and the length of 1 m. The cross-sectional area of the beam is $A = 2.0\text{E} - 4\ \text{m}^2$. The moment of inertia of the cross-section of the beam is $I_z = 2.0\text{E} - 8\ \text{m}^4$. The Poisson's ratio is $\mu = 0.3$. Now there is a harmonic sinusoidal excitation $P(t) = -p\sin(1600\pi t)N$ acting on the vertical direction of the node 3 with the initial condition $\dot{x}(0) = 0$ and $x(0) = 0$. Assume that, because of uncertainties, the Young's modulus, the mass density and the harmonic sinusoidal excitation amplitude of the cantilever beam are uncertain-but-bounded parameters, and their interval numbers are: $E^I = [194 \times 10^9, 206 \times 10^9]\ \text{N/m}^2$, $\rho^I = [97, 103]\ \text{kg/m}^3$ and $p^I = [97, 103]$ N. We also assume that the Young's modulus, the mass density and the harmonic sinusoidal excitation amplitude of the cantilever beam have all normal or Gaussian distributions in their interval numbers with the mean values (MV) $\mu_E = 200 \times 10^9\ \text{N/m}^2$, $\mu_\rho = 7800\ \text{kg/m}^3$, $\mu_p = 100$ N and the standard variances $\sigma_E = 6 \times 10^9$ N/m^2, $\sigma_\rho = 234\ \text{kg/m}^3$, $\sigma_p = 3$ N. The response regions of the fifth node in vertical direction on the cantilever beam are, respectively, computed by the interval analysis method and the probabilistic approach, and plotted in the Figs. 2 and 3. The comparison of the response region curves of the cantilever beam by the interval analysis method and the probabilistic approach is also presented in Fig. 4.

Example II. Considering a plane truss subjected to a harmonic sinusoidal excitation $P(t) = -\sin(20\pi t)N$ with the initial condition $\dot{x}(0) = 0$ and $x(0) = 0$, as shown in Fig. 5. The truss is partitioned into 6 nodes and 8 elements. Assume that the cross-sectional area of elements ①, ②, ③ and ④ are $A_1 = A_2 = A_3 = A_4 =$

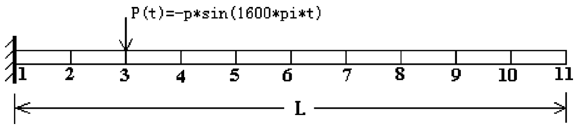

Fig. 1. A cantilever beam.

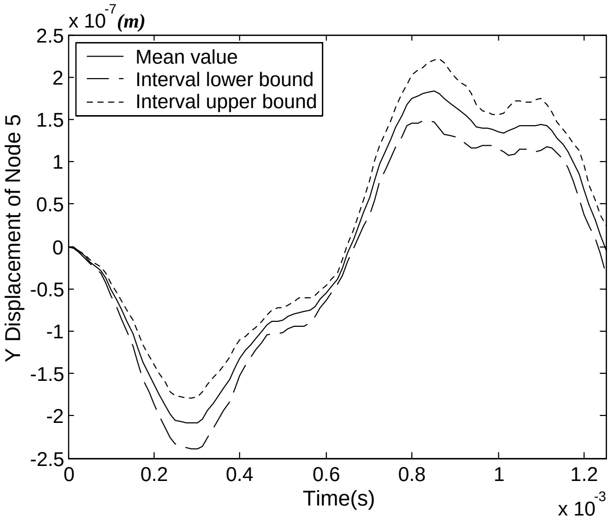

Fig. 2. Response region curves of the cantilever beam by the interval analysis method.

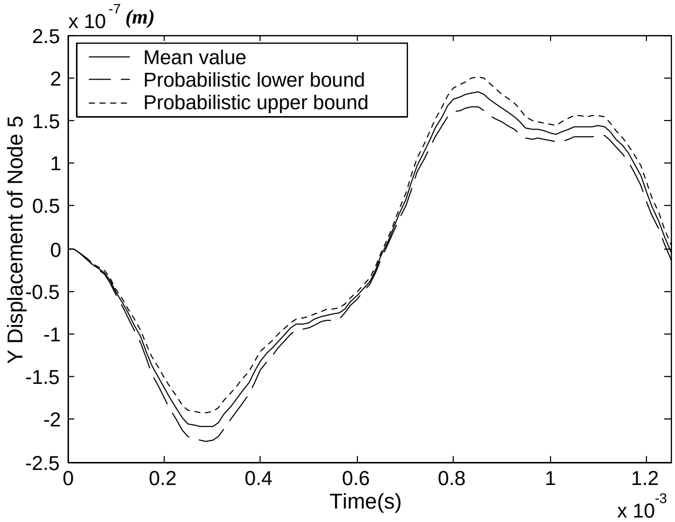

Fig. 3. Response region curves of the cantilever beam by the probabilistic approach.

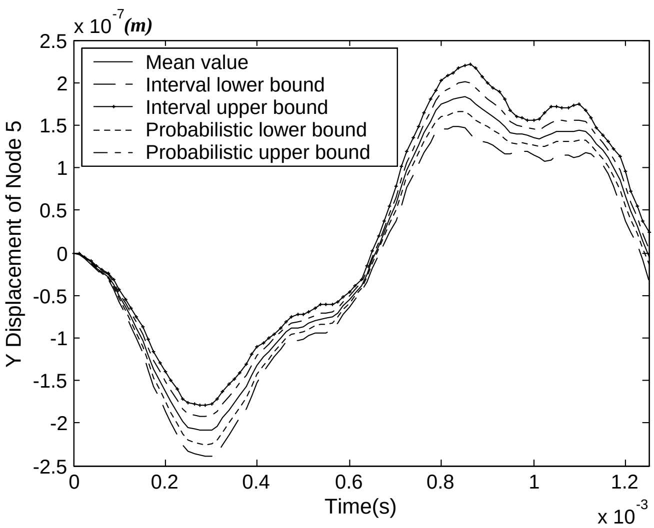

Fig. 4. Comparison of the response region curves of the cantilever beam by the interval analysis method and the probabilistic approach.

1.0×10^{-4} m^2, those for elements ⑤, ⑥, ⑦ and ⑧ are $A_5 = A_6 = A_7 = A_8 = 1.2 \times 10^{-4}$ m^2. The Poisson's ratio is $\mu = 0.3$. We assume that the Young's modulus, the mass density and the harmonic sinusoidal excitation amplitude of the plane truss are uncertain-but-bounded parameters, and their interval numbers are: $E^I = [190 \times 10^9, 210 \times 10^9]$ N/m^2, $\rho^I = [7410, 8190]$ kg/m^3 and $p^I = [95, 105]$ N. We also assume that the Young's modulus, the mass density and the harmonic sinusoidal excitation amplitude of the plane truss

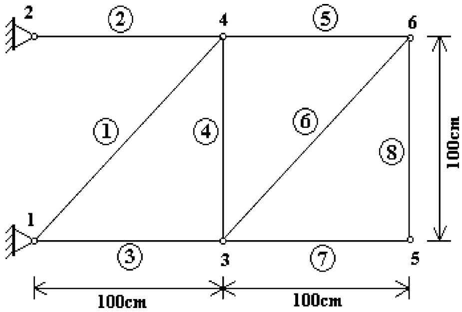

Fig. 5. A plane truss.

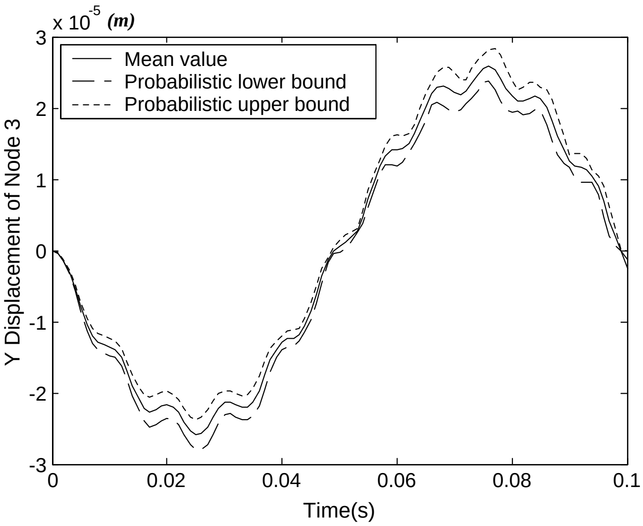

Fig. 6. Response region curves of the truss by the interval analysis method.

have all normal or Gaussian distributions in their interval numbers with the mean values (MV) $\mu_E = 200 \times 10^9$ N/m^2, $\mu_\rho = 7800$ kg/m^3, $\mu_p = 100$ N and the standard variances $\sigma_E = 10 \times 10^9$ N/m^2, $\sigma_\rho = 390$ kg/m^3, $\sigma_p = 5$ N. The response regions of the third node in vertical direction on the plane truss are, respectively, computed by the interval analysis method and the probabilistic approach, and plotted in the Figs. 6 and 7. The comparison of the response region curves of the plane truss by the interval analysis method and the probabilistic approach is also presented in Fig. 8.

From the above numerical examples, we can see that the region of the dynamical response of structures with uncertain-but-bounded parameters obtained by the interval analysis method contains that produced by the probabilistic approach. In other words, it is seen that the present interval analysis method yields larger bounds; namely, the lower bounds within the present interval analysis method are smaller than those predicted by the probabilistic approach. Likewise, the upper bounds furnished by the present interval analysis method are larger than those yielded by the probabilistic approach. This kind of results is coincident with the meaning of the probabilistic theory and interval mathematics.

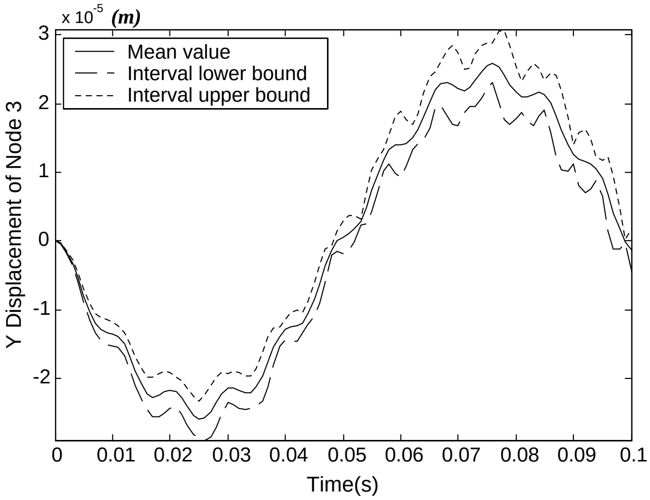

Fig. 7. Response region curves of the truss by the probabilistic approach.

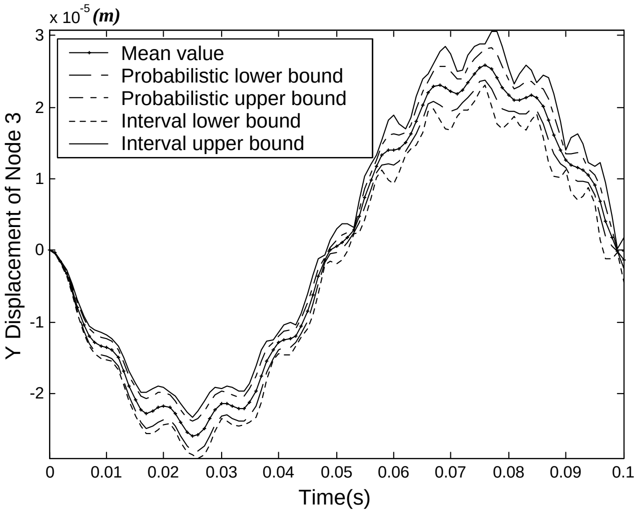

Fig. 8. Comparison of the response region curves of the truss by the interval analysis method and the probabilistic approach.

8. Conclusions

In this study we considered the dynamical response of structures with uncertain parameters. Instead of assuming extensive knowledge of the probabilistic characteristics of the uncertain parameters, we adopted a non-probabilistic, set-theoretic approach to model uncertainty in the structural parameters. In particular, we assumed that the structural parameters are uncertain-but-bounded. By finite element analysis and interval mathematics, the non-probabilistic interval analysis method for structural dynamical response is developed. Under the condition of the box or interval vector, which contains the uncertain-but-bounded parameters, determined from the probabilistic statistical information or stochastic sample test, we can also show that the width of the upper and lower bounds on the structural dynamical response yielded by the probabilistic approach is tighter than those produced by the non-probabilistic interval analysis method in the mathematical proof and the numerical examples. This kind of results is coincident with the meaning of the probabilistic theory and interval mathematics.

Acknowledgements

The work of Zhiping, Qiu was supported by the National Natural Science Foundation of the PR China and the Aeronautical Science Foundation of the PR China.

Appendix A. Determination of interval parameters based on the probabilistic statistical information

In some cases, we can obtain probabilistic statistical properties of the uncertain variables by measurements or past experience. In these cases, on the one hand, the uncertain variable should be treated as random field, on the other hand, we can determine the interval region in which the uncertain variable is varying based on these probabilistic statistical information. Before discussing some important theoretical results, we will introduce two characteristics associated with an uncertain distribution. Generally, a characteristic associated with an uncertain distribution is called a parameter. A parameter is defined to be a numerical value associated with a theoretical uncertain distribution. The two parameter we consider here are the mean value and the variance (Elishakoff, 1983; Robert et al., 1975), the former being a measure of location and the latter a measure of variation of the uncertain random variable. Since the theoretical uncertain probability distribution gives a complete description of the corresponding uncertain random variable, we will call the two parameters the mean value and the variance of the random variable or equivalently, of the probability distribution. The two parameters of a probability distribution, be it discrete or continuous, will be denoted by letters μ and D.

Suppose a random variable X can take on the values of a finite discrete set $\{x_1, x_2, \ldots, x_n\}$ according to the probability function $f(x)$. The mean value, denoted by μ, of the discrete random variable is defined by

$$\mu = E(X) = \sum_{i=1}^{n} x_i f(x_i) \tag{A.1}$$

It should be noted the mean value in the above expression is the usual weighted arithmetic average of $x_1, x_2, \ldots, x_n$. The probability function in most cases is theoretical, which is not known in general. We also say the mean value is the mathematical expectation or the expected valued of the discrete random variable X.

The second parameter of interest is the variance of the discrete random variable X, denoted by D. The variance of the discrete random variable X is defined by

$$D = E[(X - \mu)^2] = \sum_{i=1}^{n} (x_i - \mu)^2 f(x_i) \tag{A.2}$$

The variance of the discrete random variable X is simply the weighted average of $(x_1 - \mu)^2, (x_2 - \mu)^2, \ldots, (x_n - \mu)^2$, with respect to the weights $f(x_1), f(x_2), \ldots, f(x_n)$. Obviously, the variance of the discrete random variable X is also the expected value of $(X - \mu)^2$.

The mean value μ and the variance D of a continuous random variable X are given by

$$\mu = E(X) = \int_{-\infty}^{\infty} x f(x)\, \mathrm{d}x \tag{A.3}$$

and

$$D = E[(X-\mu)^2] = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\,\mathrm{d}x \tag{A.4}$$

The positive square root of the variance D is called the standard deviation of the uncertain random X and is denoted by $\sigma = \sqrt{D}$.

The parameters μ and D are referred to as the population mean value and the variance, respectively. They are generally unknown and information about them, in the form of inferences, is obtained by considering a sample from the appropriate population. This leads to the sample analogues of μ and D, which are called the sample mean value and the sample variance.

The sample mean value, which is the most important and often used in statistics, is defined by the sum of all the sample values divided by the number of observations in the sample and is denoted by

$$\mu_s = \sum_{i=1}^{n} \frac{x_i}{n} \tag{A.5}$$

where μ_s is the mean value of n values and x_i is any given value in the sample. It is an estimate of the value of the mean value of the population from which the sample was drawn. The sample mean-value is used as a measure of location is one that indicates where the center of the data is located.

Having determined the location of the data as expressed by statistics such as the mean value, the next thing to be considered is how the data are spread about these mean values. The most popular method of reporting variability is by use of the sample variance defined by

$$D_s = \frac{1}{n-1} \sum_{i=1}^{n} (X_i - \mu_s)^2 = \frac{1}{n-1} \left[\sum_{i=1}^{n} X_i^2 - n\mu_s^2 \right] \tag{A.6}$$

The sample variance is the sum of the squares of the deviations of the data points from the mean value of the sample divided by $n-1$.

The standard deviation of a sample, denoted by σ_s, is defined to be the positive square root of the sample variance, that is $\sigma_s = \sqrt{D_s}$.

Let X be a random variable with a finite variance σ_X^2 and the mean value $E(X)$. Then the following probabilistic inequality is valid, provided k is a positive integer

$$P(|X - E(X)| \geqslant k\sigma_X) \leqslant \frac{1}{k^2} \tag{A.7}$$

This inequality, named after Tchebycheff's inequality, signifies that

$$P(|X - E(X)| < k\sigma_X) > 1 - \frac{1}{k^2} \tag{A.8}$$

From the Tchebycheff's inequality, we know that the probability of the random variable X with finite variance σ_X^2 falling within k standard deviations of its mean is at least $1 - 1/k^2$. For example, for $k = 2$ we obtain

$$P(E(X) - 2\sigma_X < X < E(X) + 2\sigma_X) > \tfrac{3}{4} = 0.75 \tag{A.9}$$

for any random variable X with finite variance. For $k = 3$

$$P(E(X) - 3\sigma_X < X < E(X) + 3\sigma_X) > \tfrac{8}{9} = 0.8889 \tag{A.10}$$

For any random variable X with finite variance, the latter inequality signifies that the probability of X falling within three standard deviations of its mean value is at least 0.8889. This bound is independent of the distribution of X, provided that it has a finite variance σ_X^2. Obviously, as k increases, the probability of the uncertain variable falling within the interval $[E(X) - k\sigma_X, E(X) + k\sigma_X]$ approach unity, that is to say $X \in [E(X) - k\sigma_X, E(X) + k\sigma_X]$ is almost a certain event as k increases. For example, when $k = 10$ we have

$$P(E(X) - 10\sigma_X < X < E(X) + 10\sigma_X) > \tfrac{99}{100} = 0.99 \tag{A.11}$$

for any random variable X with finite variance σ_X^2.

Thus, if given the mean value $E(X)$ and the finite variance σ_X^2 of an uncertain variable X, we may take $[E(X) - k\sigma_X, E(X) + k\sigma_X]$, where k is sufficent large, as the interval number or vector in which the uncertain variable X is varying.

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